The Iterative Conception of Set

A Modal Reading

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The Iterative Conception of Set

Element Priority. At every stage, every set is formed from things that were formed at an earlier stage

Maximality. At every stage, *any* things formed at an earlier stages will form a set

$\emptyset \quad \{\emptyset\} \quad \{\emptyset\,\{\emptyset\}\} \quad \{\emptyset\,\{\emptyset\},\{\emptyset\,\{\emptyset\}\}\} \quad \{\emptyset\,\{\emptyset\},\{\emptyset\,\{\emptyset\}\},\{\emptyset\,\{\emptyset\},\{\emptyset\,\{\emptyset\}\}\}\},\ldots\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\},\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\},\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\},\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}\},\ldots\},\ldots$

$\emptyset \quad \{\emptyset\} \quad \{\emptyset\,\{\emptyset\}\} \quad \{\emptyset\,\{\emptyset\},\{\emptyset\,\{\emptyset\}\}\} \quad \{\emptyset\,\{\emptyset\},\{\emptyset\,\{\emptyset\}\},\{\emptyset\,\{\emptyset\},\{\emptyset\,\{\emptyset\}\}\}\},\ldots$

$\vdots$

$\vdots$

$\emptyset \quad \{\emptyset\} \quad \{\{\emptyset\}\} \quad \{\emptyset\,\{\emptyset\}\}$

$\emptyset \quad \{\emptyset\}$

$\emptyset$

The Maximally Liberal Attitude to Set Formation

- Usual to axiomatise ‘Stage Theory’ in a non-modal language.
- This suffices to recover (most of) Z.
- Non-modal Stage Theory precludes our taking the maximally liberal attitude to set formation.

Naive Comprehension$^{\Diamond}$ *Absolutely* any things can form a set

$\Box\forall xx\Diamond\exists y(xx \text{ forms } y)$

Naive Comprehension Any things do form a set

$\forall xx\exists y(xx \text{ forms } y)$

Four Ways to Understand $\Box\phi$

Tense Operators: Sets are literally created(!)

- $\Box\phi$: ‘No matter what sets are created it will always be the case that ϕ’

Context Operators: Forming sets consists in relaxing contextual restrictions on the background domain.

- $\Box\phi$: ‘No matter how the context is admissibly relaxed it will always be the case that ϕ’

Interpretation Operators: Forming sets consists in liberalising the interpretation of $\forall$.

- $\Box\phi$: ‘No matter how the interpretation is admissibly liberalised it will always be the case that ϕ’

Ontology Operators: Forming sets consists in adjusting the ‘third parameter’(!)

- $\Box\phi$: ‘No matter what objects are admissibly postulated it will always be the case that ϕ’

Languages

$L_\in$: a (non-modal) plural first-order language.

- $\neg, \rightarrow, \forall$
- Logical predicates: $=, \prec$.
- Non-logical Predicates: Set, $\in$.

$L_\in^\diamond$: a (bi-)modal plural first-order language.

- Strict Forwards Necessity Operator: $\boxdot$.
- Strict Backwards Necessity Operator: $\boxdot^{-1}$.
- Other modalities:

$$\Box^{-1}\phi =_{df} \boxdot^{-1}\phi \wedge \phi$$
$$\Box\phi =_{df} \boxdot\phi \wedge \phi$$
$$\blacksquare\phi =_{df} \Box\Box^{-1}\phi$$

Kripke Models I

A frame is a triple
$\mathscr{F} = \langle S, <, \{D_s\}_{s\in S}\rangle$

- S is a non-empty set
- $<$ is a binary relation on S
- Each D_s is a non-empty set

A model is a quintuple
$\mathscr{M} = \langle S, <, \{D_s\}_{s\in S}, \{[\![\text{Set}]\!]_s\}_{s\in S}, \{[\![\in]\!]_s\}_{s\in S} >$

- $\langle S, <, \{D_s\}_{s\in S}\rangle$ is a frame
- $[\![\text{Set}]\!]_s \subseteq D \quad (= \bigcup_s D_s)$.
- $[\![\in]\!]_s \subseteq D^2$
- $[\![=]\!]_s = \{\langle a, a\rangle : a \in D_s\}$
- $[\![\prec]\!]_s = \{\langle a, A\rangle : a \in A \subseteq D_s\}$

Kripke Models II

For $\mathscr{F} = \langle S, <, \{D_s\}_{s\in S}\rangle$ an $\mathscr{F}$-assignment α maps each singular/plural variables to a member/subset of some Dt.

$$[\![Rv_1 \ldots vv_k]\!]_s^\alpha = T \text{ iff } \langle \alpha(v_1), \ldots, \alpha(vv_k)\rangle \in [\![R]\!]_s$$
$$[\![\neg\phi]\!]_s^\alpha = T \text{ iff } [\![\phi]\!]_s^\alpha \neq T$$
$$[\![\phi \rightarrow \psi]\!]_s^\alpha = T \text{ iff } [\![\phi]\!]_s^\alpha \neq T \text{ or } [\![\psi]\!]_s^\alpha = T$$
$$[\![\forall v\phi]\!]_s^\alpha = T \text{ iff, for every } a \in D_t, [\![\phi]\!]_s^{\alpha(a/v)} = T$$
$$[\![\forall vv\phi]\!]_s^\alpha = T \text{ iff, for every } A \subseteq D_t, [\![\phi]\!]_s^{\alpha(A/vv)} = T$$
$$[\![\boxdot\phi]\!]_s^\alpha = T \text{ iff, for every } t > s, [\![\phi]\!]_t^\alpha = T$$
$$[\![\boxdot^{-1}\phi]\!]_s^\alpha = T \text{ iff, for every } t < s, [\![\phi]\!]_t^\alpha = T$$

Consequently

$$[\![\Box\phi]\!]_s^\alpha = T \text{ iff, for every } t \geq s, [\![\phi]\!]_t^\alpha = T$$
$$[\![\Box^{-1}\phi]\!]_s^\alpha = T \text{ iff, for every } t \leq s, [\![\phi]\!]_t^\alpha = T$$
$$[\![\blacksquare\phi]\!]_s^\alpha = T \text{ iff, for every } t \in S, [\![\phi]\!]_t^\alpha = T$$

(assuming that $\leq$ is connected in the last case)

Axioms I: Free Plural First Order Logic

Write $uu = vv$ for $\Box\forall t(t \prec uu \leftrightarrow t \prec vv)$, Ev for $\exists x(x = v)$, Evv for $\exists xx(xx = vv)$; and for $\phi(v_1, \ldots vv_k)$ let E_ϕ abbreviate $Ev_1 \wedge \ldots \wedge Evv_k$

(US_1)	$\forall v\phi \rightarrow (Ev \rightarrow \phi[u/v])$
(Gen_1)	If $\Gamma, Eu \vdash \phi[u/v]$ then $\Gamma \vdash \forall v\phi$
(US_2)	$\forall vv\phi \rightarrow (Evv \rightarrow \phi[uu/vv])$
(Gen_2)	If $\Gamma, Euu \vdash \phi[uu/vv]$ then $\Gamma \vdash \forall vv\phi$
(PC)	$\exists vv\forall v(v \prec vv \leftrightarrow \phi)$
(Refl)	$\forall v(v = v)$
(Subst)	$u = v \rightarrow (\phi \rightarrow \phi(u/v))$
(E_σ)	$\sigma \rightarrow E_\sigma$, for σ atomic
($\Diamond E_1$)	$\Diamond Ev$
($\Diamond E_2$)	$\Diamond Evv$

Proposition
The axiom in group I are valid on all frames.

Axioms II: Iterative Frames

An iterative frame is a frame $\mathscr{F} = \langle S, <, \{D_s\}_{s\in S}\rangle$

- $\mathscr{F}$ is (strictly) monotonic $\forall s \forall t (s < t \to D_s \subset D_t)$
- $\mathscr{F}$ has a first stage $(\exists s \in S)(\forall t \in S)(t \nless s)$
  and a limit stage $(\exists u \in S)(\forall s < u)(\exists t \in S)(s < t < u)$
  but no last stage $(\forall t \in S)(\exists u \in S)(s < u)$
- $<$ is a (strict) partial order $<$ irreflexive, transitive
  doesn't branch forwards $(\forall t \in S)(\forall u \geq t)(\forall u' \geq t)(u \leq u' \vee u' \leq u)$
  or backwards $(\forall t \in S)(\forall s \leq t)(\forall s' \leq t)(s \leq s' \vee s' \leq s)$
- $<$ is well-founded $(\forall T \subseteq S)[T \neq \emptyset \to (\exists s \in T)(\forall t \in T)(t \nless s)]$

Axioms II

$(\dot{K}^{-1})$	$\boxdot^{-1}(\phi \to \psi) \to (\boxdot^{-1}\phi \to \boxdot^{-1}\psi)$	$\boxdot(\phi \to \psi) \to (\boxdot\phi \to \boxdot\psi)$	$(\dot{K})$
$(\dot{CV}^{-1})$	$\phi \to \boxdot^{-1}\dot{\Diamond}\phi$	$\phi \to \boxdot\dot{\Diamond}^{-1}\phi$	$(\dot{CV})$
$(\dot{Nec.}^{-1})$	If $\vdash \phi$ then $\vdash \boxdot^{-1}\phi$	If $\vdash \phi$ then $\vdash \boxdot\phi$	$(\dot{Nec.})$
$(\dot{4}^{-1})$	$\boxdot^{-1}\phi \to \boxdot^{-1}\boxdot^{-1}\phi$	$\boxdot\phi \to \boxdot\boxdot\phi$	$(\dot{4})$
(Löb)	$\dot{\Diamond}^{-1}\phi \to \dot{\Diamond}^{-1}(\phi \wedge \boxdot^{-1}\neg\phi)$		
(NB^{-1})	$\Diamond\Diamond^{-1}\phi \to (\Diamond\phi \vee \Diamond^{-1}\phi)$	$\Diamond^{-1}\Diamond\phi \to (\Diamond\phi \vee \Diamond^{-1}\phi)$	(NB)
(E^{-1})	$\Diamond^{-1}\boxdot^{-1}\bot$	$\Box\dot{\Diamond}\top$	$(\not{E})$
(Lim)	$\blacklozenge(\dot{\Diamond}^{-1}\top \wedge \exists mm\, \boxdot^{-1}\dot{\Diamond}\neg Emm)$		
$(\dot{BF}_s^{-1})$	$\forall v\, \boxdot^{-1}\phi \to \boxdot^{-1}\forall v\phi$	$\boxdot\forall v\phi \to \forall v\, \boxdot\phi$	$(\dot{BF}_s)$
$(\dot{BF}_p^{-1})$	$\forall vv\, \boxdot^{-1}\phi \to \boxdot^{-1}\forall vv\phi$	$\boxdot\forall vv\phi \to \forall vv\, \boxdot\phi$	$(\dot{BF}_p)$
(Inc)	$\exists x\, \boxdot^{-1}\neg Ex$		

Proposition

The axioms in groups I and II are valid in all and only iterative frames

Elementary Consequences of Axioms II

(T)	$\Box\phi \to \phi$
(4)	$\Box\phi \to \Box\Box\phi$
(L^{+})	$\Box(\Box\phi \to \psi) \vee \Box(\Box\psi \to \phi)$
(G)	$\Box\Diamond\phi \to \Diamond\Box\phi$
	$\Diamond\Diamond^{-1}\phi \leftrightarrow \Diamond^{-1}\Diamond\phi$
	$\Box\Box^{-1}\phi \leftrightarrow \Box^{-1}\Box\phi$
(**T**)	$\blacksquare\phi \to \phi$
(**4**)	$\blacksquare\phi \to \blacksquare\blacksquare\phi$
(**B**)	$\phi \to \blacksquare\blacklozenge\phi$

Stability, Closure and Boundedness

- A formula ϕ is *absolutely stable* if (S-ϕ) and (S-$\neg\phi$) hold

  (S-ϕ) $\quad \phi \to \blacksquare\phi$
  (S-$\neg\phi$) $\quad \neg\phi \to \blacksquare\neg\phi$

- A formula ϕ is *relatively stable* if (S-ϕ^E) and (S-$\neg\phi^E$) hold.

  (S-ϕ^E) $\quad (\phi \wedge E_\phi) \to \blacksquare(E_\phi \to \phi)$
  (S-$\neg\phi^E$) $\quad (\neg\phi \wedge E_\phi) \to \blacksquare(E_\phi \to \neg\phi)$

- A formula $\phi(t)$ if *closed* if (Cl-ϕ)

  (Cl-ϕ) $\quad (\forall t : \phi)\Box\psi \to \Box(\forall t : \phi)\psi$

- A formula $\phi(t)$ is *bounded* if $\exists mm\Box\forall t(\phi(t) \to t <^{\Diamond} mm)$, and *potentially bounded* if $\Diamond\exists mm\Box\forall t(\phi(t) \to t <^{\Diamond} mm)$

Axioms III

Stability and Closure Axioms.

(S- $=^E$), (S- $\neq^E$), (S- $\prec^E$), (S- $\not\prec^E$)
(S-Set^E), (S-¬Set^E), (S- $\in^E$), (S- $\notin^E$)
(Cl- $\prec$) $\quad \forall xx(\forall t \prec xx)\Box\psi \rightarrow \Box(\forall t \prec xx)\psi$

Proposition

Whenever ϕ is stable, ϕ is bounded just in case ϕ is closed

A stable model is a model
$\langle S, <, \{D_s\}_{s \in S}, \{[\![\mathrm{Set}]\!]_s\}_{s \in S}, \{[\![\in]\!]_s\}_{s \in S} >$ such that

- $[\![\mathrm{Set}]\!]_s \cap D_t = [\![\mathrm{Set}]\!]_t \cap D_s$
- $[\![\in]\!]_s \cap D_s^2 = [\![\in]\!]_t \cap D_s^2$

Proposition

The axioms in groups I, II and III are valid in all and only stable models of iterative frames.

Talking about *absolutely* all sets

Objection. The modal reading of the iterative story deprives set theory of its intended generality.

- Set-Theorist:
  '$\forall x[\mathrm{Set}(x) \rightarrow (\exists t(t \in x) \rightarrow (\exists t \in x)(\forall y \in t)(y \notin x))]$'
- fails to express the principle of *Foundation*

Reply. We succeed in generalising over *absolutely* all sets by embedding our quantifiers within modal operators.

- $\forall x\phi$: every member of D_{s_0} satisfies ϕ
- $\Box\forall x\phi$: at every stage s, every member $D_s$satisfies ϕ
- $x \in y$: pair $\langle \alpha(x), \alpha(y) \rangle$ in $[\![\in]\!]_{s_0}$
- $\Diamond(x \in y)$: pair $\langle \alpha(x), \alpha(y) \rangle$ in $[\![\in]\!] \quad (= \bigcup_s [\![\in]\!]_s)$

Modalisation I

Modalisation. The modalisation $\phi^\Diamond$of a $L_\in$ formula ϕ is defined recursively

- $\sigma^\Diamond = \Diamond\sigma \quad$ for σ an atomic formula
- $(\neg\phi)^\Diamond = \neg\phi^\Diamond$
- $(\phi \rightarrow \psi)^\Diamond = \phi^\Diamond \rightarrow \psi^\Diamond$
- $(\forall v\phi)^\Diamond = \Box\forall v\phi^\Diamond$
- $(\forall vv\phi)^\Diamond = \Box\forall vv\phi^\Diamond$

- The intended generality of claim ϕ made in $L_\in$ is realised by its modalisation $\phi^\Diamond$
- e.g. Foundation:

'$\Box\forall x[\mathrm{Set}^\Diamond(x) \rightarrow (\Diamond\exists t(t \in^\Diamond x) \rightarrow (\Diamond\exists t \in^\Diamond x)(\Box\forall y \in^\Diamond t)(y \notin^\Diamond x))]$'

(where $\mathrm{Set}^\Diamond(x) =_{df} \mathrm{Set}(x)^\Diamond$, $\quad x \in^\Diamond y =_{df} (x \in y)^\Diamond$, etc.)

Modalisation II: Consonance with Mathematical Practice

- Set Theory is stage invariant

Theorem

Modalised formulas are absolutely stable

Corollary

The following are provably equivalent:

$$\blacksquare\phi^\Diamond, \Box^{-1}\phi^\Diamond, \Diamond^{-1}\phi^\Diamond, \phi^\Diamond, \Diamond\phi^\Diamond, \odot\phi^\Diamond, \boxdot\phi^\Diamond, \Box\phi^\Diamond$$

- Set Theorists employ classical logic

Theorem

If $\Gamma \vdash_{\mathrm{PFO}^-} \phi$, then $\Gamma^\Diamond \vdash_{\mathrm{MPFO}_f} \phi^\Diamond$.

Proposition

Plural Comprehension$^\Diamond$ holds iff $\phi^\Diamond(t)$ is potentially bounded

$$\Diamond\exists xx\Box\forall t(t \prec^\Diamond xx \leftrightarrow \phi^\Diamond(t)) \text{ iff } \Diamond\exists mm\Box\forall t(\phi^\Diamond(t) \rightarrow t \prec^\Diamond mm)$$

Extensionality$^\Diamond$

Axiom

Sets are identical if they have the same members

(Ext) $\quad \Box\forall x \Box\forall y((\Box\forall t(t \in^\Diamond x \leftrightarrow t \in^\Diamond y) \rightarrow x = y)$

Proposition

If $aa \equiv^\Diamond a$ and $bb \equiv^\Diamond b$ then

- $a \in^\Diamond b$ iff $a \prec^\Diamond bb$
- $a \subseteq^\Diamond b$ iff $aa \prec^\Diamond bb$ $\quad (=_{df} \Box\forall t(t \prec^\Diamond aa \rightarrow t \prec^\Diamond bb))$
- $a =^\Diamond b$ iff $aa =^\Diamond bb$

Element Priority

Axiom

At any stage, all sets are formed from objects that existed at some earlier stage

(EP) $\quad \blacksquare\forall x(\mathrm{Set}^\Diamond(x) \rightarrow \Diamond^{-1}\exists xx(xx \equiv^\Diamond x))$

Proposition

Whenever $\mathrm{Set}^\Diamond(a)$ exists,

- If $b \in^\Diamond a$, then $\Diamond^{-1} Eb$
- $t \in^\Diamond a$ is closed

Theorem

EP and *Extensionality* prove *Foundation*$^\Diamond$

Principle of Set Existence

Putative Axiom.

At any stage, any objects will form a set (at some later stage)

(PS∃) $\quad \blacksquare\forall xx \Diamond\exists x(\mathrm{Set}^\Diamond(x) \wedge xx \equiv^\Diamond x)$

Lemma (Naive Comprehension Schema)

Let $\phi^\Diamond(t)$ be potentially bounded, then

(NCS) $\quad (\mathrm{PS}\exists) \vdash \Diamond\exists x(\mathrm{Set}^\Diamond(x) \wedge \Box\forall t(t \in x \leftrightarrow \phi^\Diamond(t))$

Theorem

(PS∃), (EP) and *Extensionality*$^\Diamond$ prove *Empty Set*$^\Diamond$, *Pairing*$^\Diamond$, *Separation*$^\Diamond$ and *Union*$^\Diamond$

Proposition

(PS∃), (EP) and *Extensionality*$^\Diamond$ do not prove *Powerset*$^\Diamond$, or that $t \subset^\Diamond a$ is potentially bounded, or *Infinity*$^\Diamond$

Principle of Maximal Set Existence

Principle of Maximal Set Existence.

At any stage, any objects will form a set at the next stage (and thus every later stage)

($\mathrm{PS}\exists^+$) $\quad \blacksquare\forall xx \boxdot \exists x(\mathrm{Set}^\Diamond(x) \wedge xx \equiv^\Diamond x)$

Lemma

- ($\mathrm{PS}\exists^+$) proves (PS∃)
- ($\mathrm{PS}\exists^+$), (EP) and *Extensionality*$^\Diamond$ prove *Powerset*$^\Diamond$
- ($\mathrm{PS}\exists^+$), (EP) and *Extensionality*$^\Diamond$ prove *Infinity*$^\Diamond$

Theorem

($\mathrm{PS}\exists^+$), (EP) and *Extensionality*$^\Diamond$ prove all the axioms of $Z^\Diamond$

Elementary Proof Theory I

Proposition (Deduction Theorem)

If $\Gamma, \phi \vdash \psi$, then $\Gamma \vdash \phi \rightarrow \psi$

Definition

- Let L be one of the necessity operators $\blacksquare$, $\square$, $\boxdot$, $\square^{-1}$, $\boxdot^{-1}$ and M the corresponding possibility operator. Say that a set of formulas Γ is *safe for generalisation by* L if Γ proves $L\gamma$ for each $\gamma \in \Gamma$.
- Let μ be a singular or plural variable, say that Γ is *safe for generalisation by* $\forall\mu$ if no element of Γ contains the variable μ free
- Whenever Γ is safe for L (or $\forall\mu$) it is also said to be safe for M ($\exists\mu$).

Elementary Proof Theory II

Proposition (Substitution Theorem)

Let θ be a formula, and Γ a set of formulas safe for generalisation by each modal operator and quantifier in θ.

$$\text{If } \Gamma \vdash \phi \leftrightarrow \psi, \text{ then } \Gamma \vdash \theta(\phi/R) \leftrightarrow \theta(\psi/R)$$

where $\theta(\phi/R)$ is the result of uniformly substituting ϕ for predicate R.

Proposition

Let θ be a formula, O_i a modal operator or quantifier, for each $i = 1, \ldots, k$, and Γ a set of formulas safe for generalisation by $O_1, \ldots, O_k$.

$$\text{If } \Gamma \vdash \phi \rightarrow \psi, \text{ then } \Gamma \vdash O_1 \ldots O_k\phi \rightarrow O_1 \ldots O_k\psi$$