Modality and Absolute Generality

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Overview

1. The challenges to the possibility of absolutely general quantification turn on the phenomenon of *indefinite extensibility*.
2. An analysis of this phenomenon in a modal framework.
3. Two sorts of generality: *intra-world* and *inter-world*.
4. Both can withstand the challenges to absolute generality.

In favor of absolute generality

- It seems to be possible.

  The empty set has absolutely no elements.

- It appears to be incoherent to deny.

  Maybe the singularist replies that some mystical censor stops us from quantifying over absolutely everything without restriction. Lo, he violates his own stricture in the very act of proclaiming it! ([Lewis, 1991], p. 68)

Against absolute generality: range of quantification

1. When we successfully quantify, there is a determinate range of quantification.
2. This range is an object ("all-in-one principle")
3. The range is "set-like"
4. Since set theory describes all the "set-like" objects there are, the range must be a set.
5. So if absolute generality was possible, there would be a universal set.
6. Since there is no such set, absolute generality is not possible.

[Boolos, 1985], [Cartwright, 1994], [Lewis, 1991], [Rayo and Uzquiano, 1999], [Williamson, 2003]:

- The "all-in-one principle" is false.

Against absolute generality: logical consequence

1. To define logical consequence, we need to quantify over interpretations.
2. For every contentful predicate F, there is an interpretation I_F under which: 'P' is true of an object o iff o is F.
3. Interpretations are objects.
4. Consider the predicate R which is true of o iff o is not an interpretation under which P is true of o.
5. Then there is an interpretation I_R under which: 'P' is true of an object o iff o is R; that is, iff o is not an interpretation under which P is true of o.
6. So 'P' is true of I_R iff 'P' is not true of I_R.

The standard response: deny 3.

Indefinite extensibility (I)

Russell argues the paradoxes arise from the combination of two claims.

(I) **Totality**: There is a collection of all ϕ's
(II) **Extensibility**: Given any collection X of ϕ's, we can define an object $\delta(X)$ which is a new ϕ.
[That is, $\delta(X)$ is not in X but $\phi(\delta(X))$.]

The problems of Absolute Generality belong to this broader class:

	Range of Quantification	Logical Consequence
Totality	"all sets"	"all interpretations"
Extensibility	a new set	a new interpretation

Indefinite extensibility (II)

	what sort of talk permitted
HOL	many objects simultaneously
FOL	one object at a time

↓ COLLAPSE ↓

Totality has force (and HOL seems legitimate)

Extensibility has force (because Collapse does)

A modal approach to indefinite extensibility

My response to indefinite extensibility:

- Tame the collapse of HOL by adopting a modal framework.
- Put this collapse to valuable use.

Potential plural collapse

$$\Box\forall xx\,\Diamond\exists y\,\textsc{Set}(y, xx)$$

Potential conceptual collapse

$$\Box\forall F\,\Diamond\exists y\,\textsc{Ppty}(y, F)$$

Individuation as a dynamic process

1. Entities are introduced successively through a well-ordered series of stages.
2. The introduction of an entity consists in the specification of a (permanent) identity condition.
3. *Cumulativity.* The licence to individuate an object never goes away but can always be exercised at a later stage.

An associated modal logic

- Each stage is a **possible world**, which consists of the entities individuated so far, and which specifies how these entities are related.
- An **accessibility relation** $w \leq w'$) which holds iff w' is a (not necessarily proper) extension of w. So $\leq$ is
  - reflexive, anti-symmetric, and transitive.
  - directed (i.e. $x \leq y \wedge x \leq z \rightarrow \exists w(y \leq w \wedge z \leq w)$)
  - well-founded.
- The resulting **Kripke-models** validate the modal logic S4.2 = S4 + (G):

$$\Diamond\Box p \rightarrow \Box\Diamond p. \qquad \text{(G)}$$

Two sorts of generality

Actualist generality: within a given world

- Expressed by $\forall$ and $\exists$
- The arguments from indefinite extensibility are powerless as Extensibility takes us from the given world to a larger one.

Potentialist generality: across all possible worlds

- Claim: The complex strings $\Box\forall$ and $\Diamond\exists$ behave logically just like quantifiers ("modalized quantifiers").
- Claim: This generality can withstand the arguments from indefinite extensibility.

Expressing potentialist generality

Definition (Stability)

A formula $\phi(\mathbf{u})$ is stable iff the following two conditionals hold:

$$\phi(\mathbf{u}) \rightarrow \Box\phi(\mathbf{u})$$
$$\neg\phi(\mathbf{u}) \rightarrow \Box\neg\phi(\mathbf{u})$$

Theorem (Mirroring)

- *Let $\phi^\Diamond$ be the result of replacing every quantifier in a non-modal formula ϕ by the corresponding modalized quantifier.*
- *Let $\vdash^\Diamond$ be provability by $\vdash$, S4.2, and axioms stating that every atomic predicate is stable, but with no higher-order comprehension.*

Then we have:

$$\phi_1, \ldots, \phi_n \vdash \psi \quad \textit{iff} \quad \phi_1^\Diamond, \ldots, \phi_n^\Diamond \vdash^\Diamond \psi^\Diamond.$$

Indefinite extensibility in a potentialist setting

(I) **Totality**: There is a collection of all actual and possible ϕ's

(II) **Extensibility**: Given any collection X of ϕ's, it is possible to define an object $\delta(X)$ which is a new ϕ.

HOL:	pluralities	concepts
	$\downarrow$	$\downarrow$
FOL:	sets	properties

What pluralities are there?

The easy question
Any formula $\phi(u)$ *contingently* defines a plurality:

$$\exists xx\, \forall u[u \prec xx \leftrightarrow \phi(u)] \qquad \text{(Comp)}$$

The hard question
When could there be a plurality that is *necessarily* defined by $\phi^\Diamond(u)$?

$$\Diamond \exists xx\, \Box \forall u[u \prec xx \leftrightarrow \phi^\Diamond(u)] \qquad (\text{Comp}^\Diamond)$$

Answer to the hard question

- A plurality has the same elements in every possible world.
- So $\phi^\Diamond(u)$ must apply to the same objects in every possible world.

Do pluralities collapse to sets?

Actualist collapse

$$\forall xx \exists y \forall u[u \in y \leftrightarrow u \prec xx] \qquad \text{(Collapse)}$$

Potentialist collapse

$$\Box \forall xx\, \Diamond \exists y\, \Box \forall u[u \in y \leftrightarrow u \prec xx] \qquad (\text{Collapse}^\Diamond)$$

Summing up

type of quantifier	ordinary	modalized
Comprehension	✓	×
Collapse	×	✓

An application to set theory

Theorem

Assume

- *trans-world extensionality principles for pluralities and sets*
- *the modal logic S4.2*
- *potential plural collapse:* $\Box \forall xx\, \Diamond \exists y\, \textsc{Set}(y, xx)$

Then we can interpret Zermelo set theory minus Infinity and Foundation.

Theorem

Assume further that every possibility witnessed by the potential universe is witnessed by some possible world:

$$\phi^\Diamond \rightarrow \Diamond \phi \qquad (\text{Refl}_0)$$

$$\Diamond \forall x(\phi^\Diamond \rightarrow \phi) \qquad \text{(Refl)}$$

(Refl$_0$) allows us to interpret Infinity, and (Refl), also Replacement.

The road ahead

What about the semantics of languages with modalized quantifiers?

Reminder

(I) **Totality**: There is a collection of all actual and possible ϕ's

(II) **Extensibility**: Given any collection X of ϕ's, it is possible to define an object $\delta(X)$ which is a new ϕ.

The strategy

	Totality	Extensibility	
collection = set	×	✓	got
collection = property	✓	×	want

- "Grounded individuation" of concepts and properties.
- The behavior of $\phi(u)$ may only depend on entities already available.

Conclusion

Claims defended

- The problem of absolute generality is an instance of the more general problem of indefinite extensibility.
- These problems should be given a modal solution.
- Both intra-world and trans-world generality are available and (apparently) stable.

Benefits

- A new approach to iterative set theory
- New approach to the paradoxes

Costs?

- New primitive notion: modality
- Two notions of collection: sets and properties

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