Unrestricted Quantification and Model-Theoretic Semantics Based on NFU

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Workshop on Absolute Generality
Sept 8-9, 2009, ENS, Paris

Aim of talk

- I will argue that the idea of constructing a model-theoretic semantics for unrestricted quantification should not be given up.

  In particular, I will discuss:
  - Williamson's variant of Russell's Paradox
  - the All-in-One Principle (Cartwright)
- I will also argue that an extension of NFU, namely NFU + Inf + AC, is a suitable metatheory in which we may construct such a semantics.
  - Despite the fact that NF-like theories have a bad reputation,
  - and arguments to the effect that such a semantics cannot handle ordinary restriced quantification.

A few words on NF(U)

- Quine's 'New Foundations for Mathematical Logic' appeared in 1937 and the set theory presented therein became known as NF.
    - NF consists of the Axiom of Extensionality, and the Axiom Schema of Stratified Comprehension. (Hailperin (1944) showed that NF is finitely axiomatizable.)
    - NF was meant as an improvement over TST as well as over ZF.
    - In particular, sets in NF have complements and there is a universal set.
    - An intuitive model is lacking, and the consistency of NF remains an open problem.

NF(U) cont.

- In 1953 Rosser publishes *Logic for Mathematicians*, in which NF is the set theory employed.
- In 1953 Specker shows that NF $\vdash \neg$AC (and hence NF $\vdash$ Inf).
- In 1969 Jensen proves that NFU, that is, NF with urelements, is consistent relative to TST.
- In fact, NFU + Inf + AC is equiconsistent with ZF.

Structure of Williamson's argument (WA)

- **Assumption**: Interpretations are individuals (in the sense of being possible values of first-order variables).
- **Semantic Principle**: [W]hatever contentful predicate we substitute for 'F', some legitimate interpretation (say, I(F)) interprets the predicate letter P accordingly:
  For everything o, I(F) is an interpretation under which 'P' applies to o if and only if o Fs.
- **Definition**: For all o, o Rs iff o is not an interpretation under which 'P' applies to o.
- This account is (almost) verbatim from Williamson (2003).
- Substitution and instantiation then yield a contradiction.

Possible diagnoses

- *Interpretation* is an indefinitely extensible concept
- Interpretations are not individuals.
- The Semantic Principle fails.
- The definition of R fails.

Consensus

It is (quite) commonly agreed that:

(1) WA has something important to say about absolute quantification; and

(2) WA is structurally similar to Russell's paradox of classes.

However, a close look at (2) puts (1) in serious doubt; see *Williamson's barber* (Analysis, 68 (4)).

Formalisation of the language used in WA

Put:

- '$INT(x)$' for 'x is an interpretation'.
- 'P' as a name of 'P'.
- '$x \models y[z]$' for 'y is applicaple to z under x'.

Formalisation of R

- 'For all o, o Rs iff o is not an interpretation under which 'P' applies to o'

  becomes
- $\forall x(R(x) \leftrightarrow \neg(INT(x) \wedge x \models P[x]))$

Formalisation of the Semantic Principle

- The meaning of 'I' is problematic, but it is sufficient for the argument that we can quantify over (all) intepretations.
- Two possible formalisations are:
  - $\exists y \forall x((INT(y) \wedge y \models P[x]) \leftrightarrow F(x))$

    which seems to mirror Williamson's formulation of the principle; and
  - $\exists y(INT(y) \wedge \forall x(y \models P[x] \leftrightarrow F(x)))$

    which seems to capture the intended meaning of the principle.
- Each formalisation yield a contradiction by substitution and instantiation.

Russell's paradox

Russell's paradox consists of

- a reductio of the existence of an r s.t. $\forall x(x \in r \leftrightarrow x \notin x)$; and
- an argument for the existence of r (Principle of Comprehension).
- Without Comprehension we would have only the reductio of r, a situation similar in form to the pseudo-paradox of the Barber.

Structural (dis)similarity

- The semantical principle is structurally similar to the Principle of Comprehension:
  - $\exists y \forall x (x \in y \leftrightarrow \varphi(x))$
  - $\exists y \forall x (\psi(x, y) \leftrightarrow F(x))$
- But whereas the Principle of Comprehension entails the existence of r, Williamson's semantical principle does not entail the existence of R.
- In fact, if we accept the semantical principle and that interpretations are possible values of first-order variables, the argument just says that R is not a contentful predicate.

Morals

- Assumption & Semantic Principle $\Rightarrow$ R is not a contentful predicate.
- Hence, there is no interpretation corresponding to R.
- Hence, unrestricted quantification is not threatened by the argument (At least not in the absence of any argument showing that R is a contentful predicate).

Linnebo (2006) also argues that the definition of R should be rejected, but doesn't seem to think that the argument itself constitutes such a rejection, as I argue here.

The All-in-One Principle

The general principle appears to be that to quantify over certain objects is to presuppose that those objects constitute a "collection," or "completed collection"–some one thing of which those objects are the members. (Cartwright 1994, p 7)

A problem

- Cartwright notices that the adequacy of the model-theoretic characterisation of logical truth may seem to support the All-in-One Principle.
  - A sentence φ is logically true iff it is true in all $\mathscr{L}_\varphi$-models.
  - Thus, assuming I_φ to be the intended interpretation of the logical truth φ, if logical truth entails (plain) truth, i.e. truth under I_φ, then I_φ has to be one of the $\mathscr{L}_\varphi$-models.
  - I.e. I_φ ought to have some entity as its domain of quantification.
- But, Cartwright continues, despite appearences, the model-theoretic characterisation of logical truth entails plain truth even if it does not involve quantification over all interpretations.
  - Every provable sentence φ is (obviously) true.
  - By completeness of FOL, $\models \varphi$ entails $\vdash \varphi$, so φ is true.

A problem, cont.

- It has been argued that Cartwright's argument is unstable: adding Q_0 gives a logic which is not complete and hence, for such a logic, the argument fails.
- Hence, a more general conception of first-order languages, one that admits quantifiers such as Q_0, may give that the model-theoretic conception of logical truth supports the All-in-one Principle after all.
- If "support" means "imply" and the All-in-One Principle is false, this is a problem.

Falsity of All-in-One Principle

Cartwright: The All-in-One Principle is false.

> *There would appear to be every reason to think it false. Consider what it implies: [. . .] that we cannot speak of the natural numbers unless there is a set of which they are the members; [. . .] (Cartwright 1994, p. 8)*

Diagnosis

- The adequacy of model-theoretic semantics entails the All-in-One Principle.
- The All-in-One Principle entails that we cannot speak of some objects unless they constitute a set (assuming that domains are sets).
- Hence, the adequacy of model-theoretic semantics entails that we cannot speak of some objects unless they constitute a set.
- But this seems like a mistake! E.g., when doing model theory, we often quantify over all models, even though they do not consitute a set.

Semantic All-in-One

- The general principle that model-theoretic semantics appears to support is that to represent quantification over certain objects in a semantic theory is to presuppose that those objects constitute a "collection," or "completed collection"–some one thing of which those objects are the members.
- This principle is much harder to deny than the (original) All-in-One Principle!

Common criticisms of NF(U)

- I suggest that we should trade ZFC for (an extension of) NFU, and construct a model-theoretic semantics in that theory.
- But there are familiar criticisms of NF(U).

> *[. . .] all known set theories with a universal set, such as Quine's New Foundations, are not only technically unappealing but have lacked any satisfactory inuitive model or conception of the entities in question. It would therefore be folly to trade traditional ZFC for one of those alternative theories. (Linnebo 2006, p. 156)*

concerning "technically unappealing"

> *Set theory is that branch of mathematics whose task is to investigate mathematically the fundamental notions "number", "order", and "function", taking them in their pristine, simple form, and to develop thereby the logical foundations of all of arithmetic and analysis; thus it constitutes an indispensable component of the science of mathematics. (Zermelo 1908)*

Quine, when working on NF in 1936, had similar objective for set theory in mind. So why did he use NF?

"technically unappealing" cont.

Zermelo's system itself was free of both drawbacks, but in its multiplicity of axioms it seemed inelegant, artificial, and ad hoc. I had not yet appreciated how naturally his system emerges from the theory of types when we render the types cumulative and describe them by means of general variables. I came to see this only in january 1954, [. . .]. If I had appreciated it in 1936, I might not have pressed on to "New Foundations."

But I might have still. For I disliked the lack of a universe class in Zermelo's system, and the lack of complements of classes, and in general the lack of big classes. (Quine 1987)

"technically unappealing" cont.

- Also, Rosser, in his *Logic for Mathematicians*, acknowledges that he may choose between ZF and NF, and opts for the latter.
- Rosser published his work in 1953.
- Since Quine didn't come to see how naturally Zermelo's system emerges from the cumulative hierarchy until 1954, we may suspect that neither did Rosser. (This is also witnessed by the lack of considerations of intuitive models in his discussion.)
- Thus, one suspects, Rosser's main argument for preferring NF was *technical*.

On intuitive models and the conception of sets

- Dummett suggests that the lack of a model, intuitive or otherwise, equals the lack of a subject-matter.
- Interestingly he makes his claim with an eye to NF.

> *Whatever mathematicians profess, mathematical theories conceived in a wholly formalist spirit are rare. One such is Quine's New Foundation system of set theory, devised with no model in mind, but on the hunch that a purely formal restriction on the comprehension axiom would block all contradictions. The result is not a mathematical theory, but a formal system capable of serving as an object of mathematical investigation: . . .*

> *without some conception of what we are talking about, we do not have a theory, because we do not have a subject-matter. [. . .] if an angel from heaven assured us of its consistency, we should still not have a mathematical theory until we attained a grasp of the structure of a model for it. (Dummett 1991, p. 230.)*

But:

(a) We do have some grasp of the models of NFU + Inf + AC.

(b) Even disregarding (a), it may be possible to use NF(U) in a *non-formalist* way, as being *about* sets.

Dummett cont.

- Compare:

  > *That the concepts of set and being a member of obey the axiom of extensionality is a far more central feature of our use of them than is the fact that they obey any other axiom. A theory that denied, or even failed to affirm, some of the other axioms of ZF might still be called a set theory, albeit a deviant or fragmentary one. (Boolos 1971)*

- Thus, we might lack a conception of the structure of the model of our set theory, but still we may claim that it is *about* sets, in particular, the sets we use when constructing our semantics.

Models

Definition

An $\mathscr{L}$-*model* $\mathcal{M}$ is an ordered couple $\langle M, I \rangle$, where M is a nonempty set and I is a function from the vocabulary of the (first-order) language $\mathscr{L}$ such that

- $I(P^n) \subseteq M^n$, if P^n is a n-ary predicate in $\mathscr{L}$;
- $I(c) \in M$, if c is an individual constant in $\mathscr{L}$;
- $I(f^m) \in [M^m \to M]$, if f^m is an m-ary functionsymbol in $\mathscr{L}$.

A set of all models?

- Though NFU is (in)famous for accepting big collections of objects as sets, we cannot be sure that there is a set of all $\mathscr{L}$-models for some $\mathscr{L}$ which we assume to contain the predicate letter P and the individual constant c as its sole non-logical constants– the concept of being an $\mathscr{L}$-model is not stratified.
- $\mathscr{L}$-model$(z) \Leftrightarrow \exists I, M(z = (M, I) \wedge M \neq \emptyset \wedge \forall w(w \in I \rightarrow \exists x, y(w = (x, y))) \wedge \forall x, y, v((x, y) \in I \wedge (x, v) \in I \rightarrow y = v) \wedge \forall y((P, y) \in I \rightarrow y \subseteq M \wedge (c, y) \in I \rightarrow y \in M))$

Syntax

- To enable recursive definitions in the meta-language of terms, formulas and sentences of an object language we assume a gödel numbering of the vocabulary.
- The interpretability of PA in our meta-theory then allows us to employ recursive definitions in the standard way.
- Note that Inf is needed for the interpretability of PA.
- The definitions of truth (in a model), logical truth, and logical consequence are all standard.

WA again

There seem to be no reasons within NFU for accepting R (from Williamson's argument) as a contentful predicate, since its defining condition is not stratified.

- The formula $x \models P[y]$ is stratified only if the type asigned to y is one lower than the type assigned to x.
- Since the unstratifiable '$x \models P[x]$' is part of the defining condition of R, the whole condition is unstratifiable.
- Hence, we cannot conclude that R is a set from stratified comprehension.

Restricted quantification

It might be feared that our semantics cannot handle ordinary restricted quantification or logical consequence since not all sub-classes of V are sets.

(a) Some collections are not sets in NFU + Inf + AC, e.g. the Russell class, and hence we cannot represent quantification over all and only the objects in such collections.

(b) Moreover, some extensions of predicates, e.g. the extension of 'non-self-membered set', seem to be unavailable in our semantics, and hence, one may fear, our definition of logical consequence (and truth) is not adequate.

(c) Also, since NFU + Inf + AC seems to allow more models than ZF, one may suspect that we should be able to falsify

Restricted quantification cont.

The real worry concerns logical consequence.
Consider thus a (finite) first-order language with a deductive system D. It is straightforward to show that the standard proofs for completeness and soundness holds.

- (soundness) If $\Gamma \vdash_D \varphi$, then $\Gamma \models_{\text{NFU}} \varphi$.
- (completeness) If $\Gamma \models_{\text{NFU}} \varphi$, then $\Gamma \vdash_D \varphi$ (via a Henkin-model in the standard way).

Thus, there is no formula φ and set of formulas Γ such that $\Gamma \not\models_{NFU} \varphi$ and $\Gamma \models \varphi$, and vice versa. I.e. for all φ and Γ we have that $\Gamma \models_{NFU} \varphi \Leftrightarrow \Gamma \models \varphi$

Concluding remarks

- Model-theoretic semantics in NFU + Inf + AC seems to be capable of representing unrestricted quantification, i.e. quantification over V; at least for (standard) first-order langauges.
- There seems to be a possibility that, for some language of greater expressive powers than the language of FOL, we may get a new relation of logical consequence when working in NFU. Question: Does this happen, and if so, for what languages?
- Which results in abstract model theory are affected by trading ZFC for NFU + Inf + AC?